INVARIANCE OF THE APPROXIMATELY REACHABLE

SET UNDER NONLINEAR PERTURBATIONS

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Invariance of the Approximately Reachable Set Under Nonlinear Perturbations

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ABSTRACT:

Consider a linear control system of the form: $\dot{x} = \mathbf{A}x + \mathbf{B}(u)$. The approximately reachable set is the closure (in the state space X) of $\mathcal{K}_o := \{x(T) : u \epsilon \mathcal{U}_{ad}\}$. We consider perturbation by a nonlinearity giving: $\dot{x} = \mathbf{A}x + \mathbf{F}(x) + \mathbf{B}(u)$ and ask when the corresponding $\overline{\mathcal{K}}_{\mathbf{F}}$ is the same as $\overline{\mathcal{K}}_o$. The concern is to reduce this to analysis of the relation to $\mathcal{K}_o$ of $\mathcal{K}_g$, obtained from an affine perturbation: $\dot{x} = \mathbf{A}x + g + \mathbf{B}(u)$, for $g \epsilon \mathcal{G}$.

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1. Introduction

We consider a control system given by

$$(1.1) \qquad \dot{x} = \mathbf{A}x + \mathbf{B}u, \qquad u \in \mathcal{U}_{ad}$$

with $\mathbf{A}$ linear and generating a semigroup $\mathbf{S}$ – or, more generally (permitting non-autonomous $\mathbf{A}(\cdot)$), a *fundamental solution* $\mathbf{S}(t,s)$ $(0 \leq s \leq t)$.

We consider two possible forms of perturbation of the system (1.1): first

$$(1.2) \qquad \dot{x} = \mathbf{A}x + g + \mathbf{B}u, \qquad u \in \mathcal{U}_{ad}$$

with the *affine perturbation* g taken from some given set $\mathcal{G}$ of functions and second

$$(1.3) \qquad \dot{x} = \mathbf{A}x + \mathbf{F}x + \mathbf{B}u, \qquad u \in \mathcal{U}_{ad}$$

where $\mathbf{F}$ is to be a nonlinear operator of Nemytsky type:

$$(1.4) \qquad \dot{x} = [\mathbf{F}x](t) := f(t, x(t)).$$

Our object is to reduce the analysis of the effect of the *quasilinear perturbation* (1.3) on the *approximately reachable set* $\bar{K}$ to the (presumably simpler) analysis of the effect of affine perturbations (1.2) as g ranges over a set $\mathcal{G}$ for which we will have $\mathbf{F}x \in \mathcal{G}$ in (1.3). In particular, we seek conditions under which $\bar{K}$ is invariant under the perturbation: (1.1) $\longmapsto$ (1.3).

We have already investigated the corresponding invariance of the (exactly) reachable set in a sequence of papers [7], [8], [10], [11] and the work presented here represents an extension of this work in two directions: the consideration of approximate rather than exact reachability and the consideration of control sets which do not form a linear space (with linear $\mathbf{B}$). The arguments used here put this work in the setting of "the fixed point approach to controllability" and we refer to [4] and its references for further historical discussion of this approach.

2. Formulation

Let us first specify the setting for the problem. The *state space* X is to be a Banach space which, for simplicity, we take to be reflexive (although it would be sufficient to require only RNP, the Radon-Nikodym Property, cf., [5]). We assume that $\mathbf{A}(\cdot)$ generates a *fundamental solution* (evolution system) $\mathbf{S}$ – i.e.,

$$
\begin{aligned}
&(i) &&\mathbf{S}(t,s) \text{ is a bounded linear operator on } X \\
&&&\text{with } \|\mathbf{S}(t,s)\| \leq M \text{ for } 0 \leq s \leq t \leq T; \\
&(ii) &&\mathbf{S}(t,s)\mathbf{S}(s,r) = \mathbf{S}(t,r) \text{ for } 0 \leq r \leq s \leq t \leq T; \\
&(iii) &&\mathbf{S}(t,s)\xi \longrightarrow \xi \text{ as } t \to s+ \text{ for } \xi \in X; \\
(2.1) \quad &(iv) &&d\mathbf{S}(t,s)\xi/dt = \mathbf{A}(s)\xi \text{ for } t = s \text{ and } \xi \in \mathcal{D} = \mathcal{D}(\mathbf{A}(s)).
\end{aligned}
$$

This permits us to introduce the notion of a *mild solution* of (1.1) or (1.2) [6]: we set

$$(2.2) \qquad \bar{x}(t) := \mathbf{S}(t,0)x(0)$$

(This assumes an initial condition specifying $x(0) \in X$.) and define a linear map

$$(2.3) \qquad \mathbf{L} : v \mapsto y \quad \text{with} \quad y(t) := \int_o^t \mathbf{S}(t,s)v(s) \, ds$$

for suitable $v(\cdot)$ so that with $w := \mathbf{B}u$ we have:

$[\bar{x} + \mathbf{L}w]$ is the mild solution of (1.1),

$[\bar{x} + \mathbf{L}(g + w)]$ is the mild solution of (1.2).

In this formulation, (1.3) corresponds to the nonlinear integral equation (abstract Volterra equation of second kind):

$$x(t) = \bar{x}(t) + \int_o^t \mathbf{S}(t,s)[f(s,x(s)) + w(s)] \, ds$$

or, equivalently, to the operator equation:

$$(2.4) \qquad x = \bar{x} + \mathbf{L}\mathbf{F}x + \mathbf{L}w$$

with $w := \mathbf{B}u$, $u \in \mathcal{U}_{ad}$.

Until one specifies the function spaces involved this is purely formal but we note now that, although we refer for convenience to (1.1), (1.2), (1.3), we will always be interpreting 'solution' in the present sense: as 'mild solution' and through (2.4). We also make, now, our first basic observation: neither $\mathbf{B}$ nor $u \in \mathcal{U}_{ad}$ (nor their individual properties) can be relevant to any of (1.1), (1.2), (1.3), but only $w := \mathbf{B}u \in \mathcal{W}_{ad}$ where

$$(2.5) \qquad \mathcal{W}_{ad} := \{w := \mathbf{B}u \text{ for some } u \in \mathcal{U}_{ad}\}.$$

One other reduction is immediately available. We can consider $x^\dagger := x - \bar{x}$ and, defining

$$f^\dagger(t,\xi) \quad := \quad f(t,\xi + \bar{x}(t)) \quad \text{so}$$

$$(2.6) \qquad \mathbf{F}^\dagger x^\dagger := f^\dagger(\cdot, x^\dagger(\cdot)) \quad = \quad f(\cdot, x^\dagger(\cdot) + \bar{x}(\cdot)) = \mathbf{F}(x^\dagger + \bar{x}) = \mathbf{F}x,$$

we see that x is a solution of (2.4) if and only if $x^\dagger$ is a solution of: $x^\dagger = \mathbf{L}\mathbf{F}^\dagger x^\dagger + \mathbf{L}w$ — which, of course, is just (2.4) with $\bar{x} = 0$ and the modified nonlinearity. There is thus no loss of generality in taking $\bar{x} = 0$, corresponding to the specification $x(0) = 0$ as initial condition; henceforth we *do* take $\bar{x} = 0$ and simply write $\mathbf{F}$ for $\mathbf{F}^\dagger$ as above.

We now introduce (reflexive) Banach spaces V, W compatible with X in the sense that the set $V \cap X$ is dense both in V and in X with

$$v_k \in V \cap X, \quad v_k \xrightarrow{V} \bar{v}, \quad v_k \xrightarrow{X} \bar{x} \implies \bar{v} = \bar{x} \in V \cap X$$

and similarly for $[W, X]$. It will thus be possible to make suitable extensions or restrictions of $\mathbf{S}(t, s)$ so, e.g., the formal definition (2.3) may make sense for V- or W-valued functions. We set $\mathcal{X} := C([0, T] \to X)$ and introduce spaces $\mathcal{V}$ and $\mathcal{W}$ of V- and W-valued functions, respectively, on $[0, T]$. Our underlying set of 'solvability hypotheses' is:

(H_1)

    (i) $\mathbf{L} : \mathcal{W} \to \mathcal{X}$ and $\mathbf{L} : \mathcal{V} \to \mathcal{X}$ are continuous (with suitable interpretation of $\mathbf{S}(t, s)$ in (2.3)); we write $\mathbf{L}_\mathcal{W}$ or $\mathbf{L}_\mathcal{V}$ where it is desirable to make the distinction explicit.

    (ii) For each $w \in \mathcal{W}$ there is a unique $g \in \mathcal{V}$ such that

$$(2.7) \qquad g = \mathbf{F}x \quad \text{for} \quad x := \mathbf{L}_\mathcal{V} g + \mathbf{L}_\mathcal{W} w$$

so $x \in \mathcal{X}$ is the (unique) solution of (2.4).

    (iii) The well-defined map $\mathbf{G} = \mathbf{G_F} : \mathcal{W} \to \mathcal{V} : w \mapsto g$ given by (ii) is *continuous* and *compact*.

Our present formulations of (1.1), (1.2), (1.3) are, respectively:

$$(2.8) \qquad\qquad x = \mathbf{L}_\mathcal{W} w \qquad (w \in \mathcal{W}_{ad}),$$

$$(2.9) \qquad\qquad x = \mathbf{L}_\mathcal{V} g + \mathbf{L}_\mathcal{W} w \qquad (w \in \mathcal{W}_{ad}, g \in \mathcal{G}),$$

$$(2.10) \qquad x = \mathbf{L}_\mathcal{V} \mathbf{G} w + \mathbf{L}_\mathcal{W} w =: \mathbf{L_F} w \qquad (w \in \mathcal{W}_{ad})$$

where $\mathcal{W}_{ad}$ is now taken to be a (specified) subset of $\mathcal{W}$ and $\mathcal{G}$ is a specified subset of $\mathcal{V}$.

REMARK: In this paper we will work with (H_1) as an abstract hypothesis. We note, however, that [10] provides four alternate sets of more concrete conditions on $X, W, V, \mathbf{S}(\cdot\cdot), f(\cdot\cdot)$ under which one can verify (H_1). For convenience of reference we present these here, converted to our present notation. For this, we take $\mathcal{V}, \mathcal{W}$ to have the form

$$(2.11) \qquad \mathcal{V} := L^p([0, T] \to V), \qquad \mathcal{W} := L^{p'}([0, T] \to W)$$

and introduce another possible space Y compatible with X. We assume

$$1 < p, \; p' < \infty; \qquad 1 \le \bar{p} < p;$$

$$1/p + 1/q = 1/p' + 1/q' = 1; \quad 1/p + 1/\bar{q}, 1/p' + 1/\bar{q}' \le 1 + 1/\bar{p}$$

and, for non-negative scalar functions on $[0, T]$:

$$\rho_V \in L^q, \rho_W \in L^{q'}, \hat{\rho}_V \in L^{\bar{q}}, \hat{\rho}_W \in L^{\bar{q}'}, \alpha \in L^p, \rho_Y \in L^1,$$

assume that:

(C_1)

(i) $\quad \|\mathbf{S}(t,s)\|_{V \to X} \leq \rho_V(t-s), \quad \|\mathbf{S}(t,s)\|_{W \to X} \leq \rho_W(t-s),$
$\quad \|\mathbf{S}(t,s)\|_{V \to Y} \leq \rho_V(t-s), \quad \|\mathbf{S}(t,s)\|_{W \to Y} \leq \hat{\rho}_W(t-s);$

(ii) $\quad \|\mathbf{S}(t,s) - \mathbf{S}(t',s)\|_{Y \to X}, \quad \|\mathbf{S}(t,s) - S(t',s)\|_{W \to X} \leq \varepsilon$
$\quad$ for $0 \leq s \leq t' - \varepsilon, t' < t \leq T$ with $\varepsilon = \varepsilon(h) \longrightarrow 0$ as $h := t - t' \longrightarrow 0;$

(iii) $\quad |f(t,\eta)|_V \leq \alpha(t) + \beta|\eta|_Y^{\bar{r}}. \hspace{3cm} (\bar{r} := \bar{p}/p < 1),$
$\quad \|\mathbf{S}(t,s)[f(s,\eta) - f(s,\eta')]\|_Y \leq \rho_Y(t-s)|\eta - \eta'|_Y.$

To (C_1) we may adjoin:
(C_2) Let *any one* of the following hold:

(i) For some Banach space Z such that $Y \hookrightarrow Z$ is a compact embedding, assume that for $\delta > 0$ there exists M_δ such that

$$\|\mathbf{S}(t, t - \delta)\|_{Z \to Y} \leq M_\delta \qquad \text{for} \quad \delta \leq t \leq T.$$

(ii) For some Banach space Z such that $Z \hookrightarrow Y$ is a compact embedding, strengthen $(C_1 - i)$ by requiring:

$$\|\mathbf{S}(t,s)\|_{W \to Z} \leq \hat{\rho}_W(t-s) \qquad (Z \text{ replacing } Y).$$

(iii) For some Banach space Z such that $Y \hookrightarrow Z$ is a compact embedding, strengthen the growth condition in $(C_1 - iii)$ by requiring

$$|f(t,\varsigma)|_V \leq \alpha(t) + \beta|\varsigma|_Z^{\bar{r}}$$

with $\bar{r} := \bar{p}/p < 1 \quad (Z \text{ replacing } Y).$

(iv) Take $Y = X$ reflexive in (C_1); for some Banach space Z such that $X = Y \hookrightarrow Z$ is a compact embedding, assume that for each $\mu > 0$ there exists $\alpha_\mu \in L^p$ for which

$$|\varsigma|_Z \leq \mu \implies |f(t,\varsigma)|_V \leq \alpha(t).$$

THEOREM 1: Let $\mathcal{X}, \mathcal{W}, \mathcal{V}$ be as above and assume (C_1). Then one has $(H_1 - i)$ and $(H_1 - ii)$ as well as the continuity of $\mathbf{G} = \mathbf{G_F} : \mathcal{W} \to \mathcal{V}$ and the growth condition

$$(2.12) \qquad |g|_{\mathcal{V}} := |\mathbf{G}w|_{\mathcal{V}} \leq C_o + C_1|w|_{\mathcal{W}}^{\bar{r}} \quad (\bar{r} < 1).$$

If, in addition, we assume (C_2), then $\mathbf{G}$ is also compact. I.e.,

$$(C_1) + (C_2) \Longrightarrow (H_1) + (2.12).$$

PROOF: See [10] for details. We note here only that $(C_1 - ii)$ is used to give $(H_1 - i)$ and that $(C_1 - i, iii)$ give a solution of (2.4) initially in $\mathcal{Y} := L^p([0,T] \to Y)$ from which one obtains $g := f(\cdot, x(\cdot)) \in \mathcal{V}$ by Krasnoselskiĭ's Theorem [2]. These arguments are fairly standard, using convolution estimates from the form of $(C_1 - i)$. One then has $x \in \mathcal{X}$ from $x = \mathbf{L}g + \mathbf{L}w$ and $(H_1 - i)$. The four alternative arguments for compactness of $\mathbf{G}$ from (C_2) use the Aubin Compactness Theorem [1], the Arzela-Ascoli Theorem, and an argument from [9]. $\square$

3. Formulation (Continued)

In the previous section we formulated the 'dynamics' of the problem, introducing the relevant spaces and the operators $\mathbf{L}_{\mathcal{W}}, \mathbf{L}_{\mathcal{V}}, \mathbf{G}$ to obtain (2.8), (2.9), (2.10). In this section we wish to consider the various reachable and approximately reachable sets.

Let $\mathbf{E}$ be the evaluation map at the terminal time:

$$\mathbf{E} : \mathcal{X} \to X : x(\cdot) \longmapsto \xi := x(T)$$

and, for brevity, let $\mathbf{T} := \mathbf{EL}$ or, more specifically,

$$\mathbf{T}_o \quad : \quad \mathcal{W} \to X : w \longmapsto \mathbf{EL}w := [\mathbf{L}w](T),$$
$$\mathbf{T}_{\mathcal{V}} \quad : \quad \mathcal{V} \to X : g \longmapsto \mathbf{EL}g := [\mathbf{L}g](T),$$
$$(3.1) \qquad \mathbf{T_F} \quad := \quad \mathbf{T}_{\mathcal{V}}\mathbf{G} + \mathbf{T}_o : \mathcal{W} \to X : w \longmapsto \xi := \mathbf{E}x \text{ such that } (2.4).$$

Clearly, in view of $(H_1 - i)$ and the definition of $\mathcal{X}$, the linear operators $\mathbf{E}, \mathbf{T}_o, \mathbf{T}_{\mathcal{V}}$ are continuous and with $(H_1 - iii)$ so is $\mathbf{T_F}$. The (exactly) *reachable sets* for (1.1),(1.2), and (1.3) are then $\mathcal{K}_o := \mathcal{K}_o(\mathcal{W}_{ad}), \mathcal{K}_g := \mathcal{K}_g(\mathcal{W}_{ad})$, and $\mathcal{K}_{\mathbf{F}} := \mathcal{K}_{\mathbf{F}}(\mathcal{W}_{ad})$, respectively, where for subsets $\mathcal{W}_* \subset \mathcal{W}$ we define:

$$\mathcal{K}_o(\mathcal{W}_*) \quad := \quad \mathbf{T}\mathcal{W}_* := \{\mathbf{EL}w : w \in \mathcal{W}_*\},$$
$$\mathcal{K}_g(\mathcal{W}_*) \quad := \quad \{\mathbf{E}(\mathbf{L}g + \mathbf{L}w) : w \in \mathcal{W}_*\} = \mathbf{T}g + \mathcal{K}_o(\mathcal{W}_*),$$
$$(3.2) \qquad \mathcal{K}_{\mathbf{F}}(\mathcal{W}_*) \quad := \quad \{\mathbf{E}x : (2.4) \text{ for } w \in \mathcal{W}_*\} = \{\mathbf{T_F}w : w \in \mathcal{W}_*\}.$$

The approximately reachable sets for (1.1), (1.2), (1.3) are then the corresponding X-closures: $\bar{\mathcal{K}}_o, \bar{\mathcal{K}}_g, \bar{\mathcal{K}}_{\mathbf{F}}$, respectively (or $\bar{\mathcal{K}}_o(\mathcal{W}_{ad})$, etc.). We will set

$$\mathcal{G}_* \quad := \quad \mathbf{G_F}\mathcal{W}_{ad} := \{g \in \mathcal{V} : g = \mathbf{G}w \text{ for some } w \in \mathcal{W}_{ad}\},$$
$$(3.3) \qquad \mathcal{V}_a \quad := \quad \{g \in \mathcal{V} : \bar{\mathcal{K}}_g = \bar{\mathcal{K}}_o\} = \{g \in \mathcal{V} : \mathbf{T}g + \bar{\mathcal{K}}_o = \bar{\mathcal{K}}_o\}.$$

Note that $\mathcal{V}_a = \mathcal{V}_a(\mathcal{W}_{ad})$ does not depend on $\mathbf{F}$. Our basic reachability hypothesis will be

$$(H_2) \quad \mathcal{G}_* \subset \mathcal{V}_a, \quad \text{i.e.,} \quad \bar{K}_g = \bar{K}_o \quad \text{for each} \quad g = \mathbf{G}w \quad (w \in \mathcal{W}_{ad}),$$

which is easily seem to be equivalent to

$$(3.4) \qquad \xi + \mathbf{T}g, \ \xi - \mathbf{T}g \in \bar{K}_o \quad \text{for each} \quad \xi \in K_o, \ g \in \mathcal{G}_*.$$

We note that when $\bar{K}_o = -\bar{K}_o$ (e.g., if $w \in \mathcal{W}_{ad} \implies -w \in \mathcal{W}_{ad}$) we need only check that $K_g = \mathbf{T}g + K_o \subset \bar{K}_o$ for each $g \in \mathcal{G}_*$ and that when $\mathcal{W}_{ad}$ is the whole space $\mathcal{W}$ the condition (3.4) reduces to a range inclusion: $\mathcal{R}(\mathbf{T}_\mathcal{V}\mathbf{G}) \subset \overline{\mathcal{R}(\mathbf{T}_o)}$.

Actually, we will use the inclusion

$$(3.5) \qquad K_g \subset \bar{K}_o \quad \text{for each} \quad g \in \mathcal{G}_*(\text{ i.e.,} \quad \xi + \mathbf{T}g \in \bar{K}_o \quad \text{for} \quad \xi \in K_o, g \in \mathcal{G}_*)$$

but will be forced to strengthen the reverse inclusion $K_o \subset \bar{K}_g$ (i.e., $\xi - \mathbf{T}g \in \bar{K}_o$ for $\xi \in K_o$, $g \in \mathcal{G}_*$) to obtain the desired invariance result: $\bar{K}_\mathbf{F} = \bar{K}_o$. Half of this result is easy.

LEMMA 1: Assume $(H_1 - ii)$ and (3.5). Then $\bar{K}_\mathbf{F} \subset \bar{K}_o$.

PROOF: Clearly it is sufficient to show $\xi \in \bar{K}_o$ for each $\xi = \mathbf{T}_\mathbf{F}\ w \in K_\mathbf{F}$. We have $\xi = \mathbf{E}x$ with $x = \mathbf{L}\mathbf{F}x + \mathbf{L}w$ for some $w \in \mathcal{W}_{ad}$ so, setting $g := \mathbf{G}w$, we have $x = \mathbf{L}g + \mathbf{L}w$ so $\xi = \mathbf{T}g + \mathbf{T}w \in K_g$. By (3.5) we have $\xi \in \bar{K}_o$ as desired. $\square$

Our efforts, then, must go to showing: $K_o \subset \bar{K}_\mathbf{F}$.

For comparison we note that our previous work obtained essentially the result:

THEOREM 2: Let $\mathcal{W}_{ad}$ be the whole space $\mathcal{W}$. Assume (H_1) and (2.12); assume that $\mathcal{R}(\mathbf{T}_\mathcal{V}) \subset \mathcal{R}(\mathbf{T}_o)$ [equivalent to: $K_g = K_o$ for each $g \in \mathcal{V}$]. Then $K_\mathbf{F} = K_o$, i.e., the exactly reachable set is then invariant under the perturbation by $\mathbf{F}$.

PROOF: We give only a very brief sketch here. Following an argument much as for Lemma 1, the proof (fixing $\xi \in K_o$) showed that $\xi \in K_\mathbf{F}$ by demonstrating existence of a fixpoint for the map: $\mathcal{W} \to \mathcal{W}$ given by

$$(3.6) \qquad \mathbf{C}\mathbf{G} : w \longmapsto g := \mathbf{G}w \longmapsto \hat{w} := \mathbf{C}g$$

where the map $\mathbf{C} = \mathbf{C}_\xi : \mathcal{V} \to \mathcal{W}$ was such that

$$(3.7) \qquad \hat{w} = \mathbf{C}g \implies \mathbf{T}g + \mathbf{T}\hat{w} = \xi.$$

Once one has shown that $\mathbf{C}$ is continuous and that there is a suitable (bounded, closed, convex) set in $\mathcal{W}_{ad} = \mathcal{W}$ invariant under $\mathbf{C}\mathbf{G}$, then the Schauder Fixpoint Theorem applies to give the desired result. Here, one obtains $\mathbf{C}$ of linear growth since $\mathcal{W}_{ad}$ is the whole space. (It is an important point that $\mathbf{C}$ depends only on the range inclusion $\mathcal{R}(\mathbf{T}_\mathcal{V}) \subset \mathcal{R}(\mathbf{T}_o)$ and not at all on $\mathbf{F}$.) Thus, existence of an invariant ball follows from (2.12). $\square$

Our intention here is to use an essentially similar argument. We make the second basic observation: for present purposes we may not only fix ξ (arbitrary $\xi \in \mathcal{K}_o$) but also ε (arbitrary $\varepsilon > 0$) and, taking $\mathbf{C} = \mathbf{C}_{\xi,\varepsilon}$ in (3.7), weaken (3.7) to require only that

$$(3.8) \qquad \hat{w} = \mathbf{C}g \implies \hat{w} \in \mathcal{W}_{ad}, \quad |\xi - [\mathbf{T}g + \mathbf{T}\hat{w}]|_X \le \varepsilon.$$

One easily sees that the existence of such $\hat{w} \in \mathcal{W}_{ad}$ (for each $\xi \in \mathcal{K}_o, \varepsilon > 0$) is precisely equivalent to the inclusion $\mathcal{K}_o \subset \bar{\mathcal{K}}_g$ which forms part of our basic hypothesis (H_2); given (H_2) and the compactness of $\mathbf{G}$ it is not too difficult to construct $\mathbf{C} = \mathbf{C}_{\xi,\varepsilon}$ continuous giving (3.8) for each $g \in \mathcal{G}_*$. Unfortunately, without strengthening the condition $\mathcal{K}_o \subset \bar{\mathcal{K}}_g$ we cannot obtain a growth rate for $\mathbf{C} = \mathbf{C}_{\xi,\varepsilon}$ which, with (2.12), gives a bounded invariant set (Under the strong assumption that $\mathbf{E}\mathcal{G}_*$ is precompact in X we do, however, have Theorem 6, below.)

A second difficulty is that the Schauder Theorem requires that the invariant set be closed and convex and we would prefer to admit the possibility that $\mathcal{W}_{ad}$ be neither closed nor convex. We may be able to escape the necessity of imposing such a condition directly on $\mathcal{W}_{ad}$. We make the third basic observation: that if we can find any set $\mathcal{W}' \subset \mathcal{W}$ such that

$$(3.9) \qquad \begin{array}{ll} (i) & \mathcal{K}_{\mathbf{F}}(\mathcal{W}') \subset \mathcal{K}_{\mathbf{F}}(\mathcal{W}_{ad}) = \bar{\mathcal{K}}_{\mathbf{F}}, \\ (ii) & \xi \in \bar{\mathcal{K}}_{\mathbf{F}}(\mathcal{W}') \end{array}$$

then $\xi \in \bar{\mathcal{K}}_{\mathbf{F}}$. Thus, if we could find such a set $\mathcal{W}'$ for each $\xi \in \mathcal{K}_o$, then we would have $\mathcal{K}_o \subset \bar{\mathcal{K}}_{\mathbf{F}}$ so, with Lemma 1, we would have the desired invariance $\bar{\mathcal{K}}_{\mathbf{F}} = \bar{\mathcal{K}}_o$. In proving (3.9 -ii) it is convenient, as noted, to have $\mathcal{W}'$ bounded, closed, and convex (but possibly dependent on ξ).

The boundedness will depend on restriction to an invariant ball but we are led to ask when we can find a (large enough) closed convex set $\mathcal{W}'$ for which (3.9 -i) holds. We note from [12] a setting in which we would have (3.9 -i) for $\mathcal{W}' = \overline{co}\mathcal{W}_{ad}$ (closed convex hull in $\mathcal{W}$). For any set $\mathcal{W}_o$ of functions on $[0,T]$ we say "$\mathcal{W}_o$ has the *segment property*" if

(SP) $\qquad w_o, w_1 \in \mathcal{W}_o \implies w_s \in \mathcal{W}_o$ for each $s \in (0, T)$

where we define $w_s := \{w_1$ on $[0, s); w_o$ on $[s, T]\}$.

(Actually, this can be modified to ask only that $w_s \in \mathcal{W}_o$ for a dense set of s.)

THEOREM 3: Let $\mathcal{W}$ have the form $L^p([0,T] \to W)$ with $1 \le p < \infty$; suppose W has the Radon-Nikodym property (e.g., W any reflexive Banach space). Define $\mathbf{L}$ as above, using (2.1) and (2.3); assume (H_1). Then for any $\mathcal{W}_o \subset \mathcal{W}$ we have $\bar{\mathcal{K}}_o(\mathcal{W}_1) = \bar{\mathcal{K}}_o(\mathcal{W}_o)$ and $\bar{\mathcal{K}}_{\mathbf{F}}(\mathcal{W}_1) = \bar{\mathcal{K}}_{\mathbf{F}}(\mathcal{W}_o)$ with $\mathcal{W}_1 := \overline{co}\mathcal{W}_o$.

PROOF: We refer to [12] for the details of the proof but comment on some considerations in the argument. The setting in [12] takes $\mathbf{S}(\cdot\cdot)$ to be a semigroup (i.e., $\mathbf{A}$ autonomous in (1.1), etc.) but one easily sees that the slightly greater generality of (2.1) causes no problems.

Most of the work in [12] went to show that: for any $w, w' \in \mathcal{W}_o$ one can construct, using (SP), a sequence $\{w_k\}$ in $\mathcal{W}_o$ for which $w_k \rightharpoonup \hat{w} := (w + w')/2$ while $\mathbf{L}w_k \to \mathbf{L}\hat{w}$ in $\mathcal{X}$. The specific assumptions imposed on $\mathbf{F}$ in [12] are unnecessary in the presence of (H_1): the compactness of $\mathbf{G}$ now gives $g_k := \mathbf{G}w_k \to \hat{g}$ in $\mathcal{V}$ (for a subsequence) so $\mathbf{L}g_k \to \mathbf{L}\hat{g}$ in $\mathcal{X}$. Thus, $x_k := \mathbf{L}g_k + \mathbf{L}w_k \to \hat{x} := \mathbf{L}\hat{g} + \mathbf{L}\hat{w}$ and $g_k = \mathbf{F}x_k \to \mathbf{F}\hat{x}$ where $\hat{g} = \mathbf{G}\hat{w}$. It follows that $\hat{\xi} := \mathbf{T_F}\hat{w}$ is the limit of $\{\mathbf{T_F}w_k\}$ so $\hat{\xi} \in \bar{K}_{\mathbf{F}}(\mathcal{W}_o)$ whenever $\hat{w} = (w + w')/2$ for $w, w' \in \mathcal{W}_o$. Repeating this gives $\mathbf{T_F}\hat{w} \in \bar{K}_{\mathbf{F}}(\mathcal{W}_o)$ for any $\hat{w}$ in $\mathcal{W}' := \{$ finite convex combinations of $\mathcal{W}_o$ with binary rational coefficients$\}$ and, since $\mathcal{W}'$ is dense in $\overline{co}\mathcal{W}_o =: \mathcal{W}_1$ and $\mathbf{T_F}$ is continuous from $\mathcal{W}$ to $\mathcal{X}$, this gives $\bar{K}_{\mathbf{F}}(\mathcal{W}_1) \subset \bar{K}_{\mathbf{F}}(\mathcal{W}_o)$ as desired. Considering $\mathbf{F} = 0$ gives $\bar{K}_o(\mathcal{W}_1) = \bar{K}_o(\mathcal{W}_o)$ as well. $\square$

We finish this section with some observations about the structure of the set $\mathcal{V}_a$.

LEMMA 2: $\mathcal{V}_a$ is closed under addition and subtraction.

PROOF: Suppose $g, g' \in \mathcal{V}_a$. We first wish to show that $K_{\bar{g}} \subset \bar{K}_o$ where $\bar{g} := g - g'$, i.e., for any $\bar{\xi} \in K_{\bar{g}}$ and $\varepsilon > 0$ that there exists $\bar{w} \in \mathcal{W}_{ad}$ with $|\bar{\xi} - \mathbf{T}\bar{w}| \leq \varepsilon$. To start, we have $\bar{\xi} = \mathbf{T}\bar{g} + \mathbf{T}w_o = \xi_1 - \mathbf{T}g'$ with $\xi_1 := \mathbf{T}g + \mathbf{T}w_o$. Since $\xi_1 \in K_g$ and $g \in \mathcal{V}_a$ gives $K_g \subset \bar{K}_o$, there must be $w_1 \in \mathcal{W}_{ad}$ such that $|\xi_1 - \mathbf{T}w_1| \leq \varepsilon/2$. Now $\mathbf{T}w_1 \in K_o$ and $g' \in \mathcal{V}_a$ gives $K_o \subset \bar{K}_{g'} = \mathbf{T}_{g'} + \bar{K}_o$ so $(\mathbf{T}w_1 - \mathbf{T}g') \in \bar{K}_o$ and there must be $\bar{w} \in \mathcal{W}_{ad}$ with $|[\mathbf{T}w_1 - \mathbf{T}g'] - \mathbf{T}\bar{w}| \leq \varepsilon/2$. Since $\bar{\xi} - \mathbf{T}\bar{w} = (\xi_1 - \mathbf{T}w_1) + (\mathbf{T}w_1 - [\mathbf{T}g' + \mathbf{T}\bar{w}])$, this gives $|\bar{\xi} - \mathbf{T}\bar{w}| \leq \varepsilon$ as desired so $\bar{\xi} \in \bar{K}_o$. This shows $\mathbf{T}\bar{g} + K_o \subset \bar{K}_o$ for $\bar{g} = g - g'$. Reversing the roles of g, g' gives $-\mathbf{T}\bar{g} + K_o \subset \bar{K}_o$ or $K_o \subset \bar{K}_{\bar{g}}$. Combining gives $\bar{K}_o = \bar{K}_{\bar{g}}$ so $\bar{g} \in \mathcal{V}_a$ for $\bar{g} = g - g' \in \mathcal{V}_a - \mathcal{V}_a$, i.e., $\mathcal{V}_a$ is closed under subtraction. Trivially, $0 \in \mathcal{V}_a$ so $g' \in \mathcal{V}_a$ gives $-g' \in \mathcal{V}_a$ whence $\bar{g} = g - (-g') = g + g'$ is in $\mathcal{V}_a$ for $g, g' \in \mathcal{V}_a$, i.e., $\mathcal{V}_a$ is closed under addition also. $\square$

Note that closure under addition shows that $\mathcal{V}_a$ is always unbounded (except for the trivial case: $\mathcal{V}_a = \{0\}$) so $\mathcal{W}_{ad}$ must also be unbounded.

LEMMA 3: Suppose $\bar{K}_o$ is convex. Then $\mathcal{V}_a$ is a (closed) subspace of $\mathcal{V}$.

PROOF: We need only show that $\mathcal{V}_a$ is convex. Since we always take $\mathbf{T}_\mathcal{V}$ continuous it is obvious that $\mathcal{V}_a$ is closed in $\mathcal{V}$ and, with convexity, Lemma 2 shows $\mathcal{V}_a$ is a subspace.

Suppose, then, $\bar{g}$ is any convex combination of $\mathcal{V}_a$ so $\bar{g} = \Sigma c_j g_j$ with $c_j > 0, \Sigma c_j = 1, g_j \in \mathcal{V}_a$. For any $\xi \in K_o$ we have $\mathbf{T}\bar{g} + \xi = \Sigma c_j(\mathbf{T}g_j + \xi)$. As each $g_j \in \mathcal{V}_a$ we have each $(\mathbf{T}g_j + \xi) \in \bar{K}_o$ so convexity of $\bar{K}_o$ gives $(\mathbf{T}\bar{g} + \xi) \in \bar{K}_o$. This, for each $\xi \in K_o$, gives $K_{\bar{g}} \subset \bar{K}_o$. By Lemma 2 we have also $-\bar{g} = \Sigma c_j(-g_j)$ a convex combination of $\mathcal{V}_a$ so $[\mathbf{T}(-\bar{g}) + \xi] \in \bar{K}_o$ for each $\xi \in K_o$, i.e., $\xi \in [\mathbf{T}\bar{g} + \bar{K}_o] = \bar{K}_{\bar{g}}$. Combining gives $\bar{K}_o = \bar{K}_{\bar{g}}$ so $\bar{g} \in \mathcal{V}_a$. $\square$

Note that $\bar{K}_o$ will certainly be convex if $\mathcal{W}_{ad}$ is convex or if (e.g., under the hypotheses of Theorem 3) there is any convex $\mathcal{W}_1$ with $\bar{K}_o(\mathcal{W}_1) = \bar{K}_o$.

4. Invariance

As noted above, we will rely on Lemma 1 to show $\bar{K}_{\mathbf{F}} \subset \bar{K}_o$ and will use a fixpoint approach based on (a strengthened form of) the condition:

$$(H'_2) \qquad K_o \subset \mathbf{T}g + \bar{K}_o(\mathcal{W}') \text{ for each } g \in \{\mathbf{G}w : w \in \mathcal{W}'\} =: \mathcal{G}',$$

for some closed, convex set $\mathcal{W}' \subset \mathcal{W}$ with $K_{\mathbf{F}}(\mathcal{W}') \subset \bar{K}_{\mathbf{F}}$ in order to obtain the reverse inclusion $\bar{K}_o \subset \bar{K}_{\mathbf{F}}$ and so to obtain the desired invariance: $\bar{K}_{\mathbf{F}} = \bar{K}_o$. Our first assumption will be:

(4.1)
$$\text{There is a (closed) convex set } \mathcal{W}' \subset \mathcal{W}$$
$$\text{such that } K_o \subset \bar{K}_o(\mathcal{W}') \text{ and } K_{\mathbf{F}}(\mathcal{W}') \subset \bar{K}_{\mathbf{F}}.$$

Note that we have (4.1) with $\mathcal{W}' := \overline{co}(\mathcal{W}_{ad})$ under the hypotheses of Theorem 3. Note that the hypotheses (H'_2) just means that the set

$$(4.2) \qquad C(g) = C(g; \xi, \varepsilon) := \{w \in \mathcal{W}' : |\xi - [\mathbf{T}g + \mathbf{T}w]|_X \leq \varepsilon\}$$

is nonempty for each $g \in \mathcal{G}' := \mathbf{G}\mathcal{W}'$ (for fixed $\xi \in K_o$, $\varepsilon > 0$); let

$$(4.3) \qquad \nu(g) = \nu(g; \xi, \varepsilon) := inf\{|w|_{\mathcal{W}} : w \in C(g; \xi, \varepsilon)\}.$$

¿From (H'_2) we have $\nu(g) < \infty$ for each such g, ξ, ε but, if we set

$$(4.4) \qquad \beta(R) = \beta(R; \xi, \varepsilon) := sup\{\nu(\mathbf{G}w; \xi, \varepsilon) : w \in \mathcal{W}', |w|_{\mathcal{W}} \leq R\}$$

for $R > 0$, then this might conceivably be infinite. In terms of this, however, we can impose a condition under which the desired fixpoint argument will be available.

LEMMA 4: Assume (H_1) and (H'_2) with $\mathcal{W}'$ as in (4.1). Fix $\xi \in K_o$, $\varepsilon > 0$ and assume there is some $R = R(\xi, \varepsilon)$ such that $\beta(R; \xi, \varepsilon) < R$. Then there is some $\bar{w} \in \mathcal{W}'$ (with $|\bar{w}|_{\mathcal{W}} \leq R$) such that $|\xi - \mathbf{T}_F\bar{w}| \leq 2\varepsilon$.

PROOF: Let $\mathcal{W}_R$ be the closed convex set $\{w \in \mathcal{W}' : |w|_{\mathcal{W}} \leq R\}$ and let $\mathcal{G}_R := \{\mathbf{G}w : w \in \mathcal{W}_R\}$. Note that (i) $\mathcal{G}_R$ is precompact in $\mathcal{V}$ by $(H_1 - iii)$ and (ii) $C_R(g) := \mathcal{W}_R \cap C(g; \xi, \varepsilon)$ is nonempty for each $g \in \mathcal{G}_R$ since $\nu(g) \leq \beta(R) < R$. By (i), we can find a finite set $\{g_j : j = 1, \ldots, J\}$ such that $min_j\{|g - g_j|\} \leq \delta$ for each $g \in \mathcal{G}_R$ where $\delta := \varepsilon/2\|\mathbf{T}_{\mathcal{V}}\|$ and by (ii), we can find $w_j \in C_R(g_j)$ for each j. A standard construction gives a continuous partition of unity subordinate to the covering of $\mathcal{G}_R$ by 2δ-balls centered at $\{g_j\}$, i.e., continuous scalar functions φ_j on $\mathcal{V}$ such that

$$\varphi_j \geq 0, \ \Sigma\varphi_j(g) = 1 \text{ for } g \in \mathcal{G}_R, \qquad \varphi_j(g) > 0 \implies |g - g_j| \leq 2\delta.$$

We now define $\mathbf{C} = \mathbf{C}_{\xi, \varepsilon}$ by
$$(4.5) \qquad \mathbf{C}g := \Sigma\varphi_j(g)w_j.$$

Since $\mathcal{W}_R$ is convex, this gives $\mathbf{C} : \mathcal{G}_R \to \mathcal{W}_R$. Clearly $\mathbf{C}$ is continuous and a simple computation gives (3.9) with ε replaced by 2ε, i.e.,

$$(4.6) \qquad |\xi - [\mathbf{T}g + \mathbf{T}\mathbf{C}g]| \leq 2\varepsilon \quad \text{for every } g \in \mathcal{G}_R.$$

¿From $(H_1 - iii)$ we have $\mathbf{C}\mathbf{G} : \mathcal{W}_R \to \mathcal{W}_R$ continuous and compact so, applying the Schauder Fixpoint Theorem, there is a fixpoint $\bar{w} \in \mathcal{W}_R$, i.e., we have $\mathbf{C}\bar{g} = \bar{w}$ for $\bar{g} = \mathbf{G}\bar{w}$. Putting $g = \bar{g}$ gives $\mathbf{T}g + \mathbf{T}\mathbf{C}g = \mathbf{T}\mathbf{G}\bar{w} + \mathbf{T}\bar{w} =: \mathbf{T_F}\bar{w}$ so (4.6) gives $|\xi - \mathbf{T_F}\bar{w}| \leq 2\varepsilon$ as desired. $\square$

THEOREM 4: Assume (H_1), (3.5), and (H_2') with $\mathcal{W}'$ as in (4.1). Suppose, for each $\xi \in \mathcal{K}_o$, $\varepsilon > 0$, one were to have $\beta(R; \xi, \varepsilon) < R$ for some $R = R(\xi, \varepsilon)$. Then the approximately reachable set is invariant under the nonlinear perturbation $\mathbf{F}$, i.e., $\bar{\mathcal{K}}_{\mathbf{F}} = \bar{\mathcal{K}}_o$.

PROOF: This is an immediate corollary of Lemma 4. One obtains (fixing ξ, ε) some $\bar{w} = \bar{w}_\varepsilon \in \mathcal{W}'$ such that $|\xi - \mathbf{T_F}\bar{w}_\varepsilon| \leq 2\varepsilon$. This, for each $\varepsilon > 0$, gives $\xi = \lim_\varepsilon \mathbf{T_F}\bar{w}_\varepsilon \in \bar{\mathcal{K}}_{\mathbf{F}}$. That, for each $\xi \in \mathcal{K}_o$, gives $\mathcal{K}_o \subset \bar{\mathcal{K}}_{\mathbf{F}}$ so $\bar{\mathcal{K}}_o \subset \bar{\mathcal{K}}_{\mathbf{F}}$. Applying Lemma 1 gives the reverse inclusion. $\square$

COROLLARY: Assume $(H_1), (2.12), (3.5)$, and (H_2') with $\mathcal{W}'$ as in (4.1). Suppose, for each $\xi \in \mathcal{K}_o$ and $\varepsilon > 0$, one had a growth rate

$$(4.7) \qquad \nu(g) \leq \tilde{C}_o + \tilde{C}_1 |g|_\mathcal{V}^{\tilde{r}} \qquad \text{for } g \in \mathcal{G}'$$

where $\tilde{C}_o, \tilde{C}_1, \tilde{r}$ depend on ξ, ε but always with $\tilde{r} < 1/\bar{r}$. Then one has invariance: $\bar{\mathcal{K}}_{\mathbf{F}} = \bar{\mathcal{K}}_o$.

PROOF: Substituting (2.12) in (4.7) gives

$$\beta(R) \leq \tilde{C}_o + \tilde{C}_1 [C_o + C_1 R^r]^{\tilde{r}} = O(R^{r\tilde{r}}) = o(R)$$

as $\bar{r}\tilde{r} < 1$. Hence one can always find $R = R(\xi, \varepsilon)$ for which $\beta(R) < R$ so the Theorem applies. $\square$

The difficulty with this, of course, is that one is unlikely to be able to verify a condition such as (4.7) to enable one to restrict attention to some $\mathcal{W}_R$. There are, however, certain cases in which one can proceed.

Since we only consider $\nu(g; \xi_o, \varepsilon)$ for $\xi_o \in \mathcal{K}_o$ so $\xi_o = \mathbf{T}w_o$, we can introduce

$$\bar{\nu}(\bar{\xi}; \varepsilon) := \inf\{|w'|_\mathcal{W} : w' \in [w_o - \mathcal{W}_{ad}], \quad |\bar{\xi} - \mathbf{T}w'| \leq \varepsilon\}$$

and have $\nu(g; \mathbf{T}w_o, \varepsilon) = \bar{\nu}(\mathbf{T}g, \varepsilon)$. Observe that if we consider $\mathcal{W}_{ad} = \mathcal{W}$, then scaling gives $\bar{\nu}(\lambda\bar{\xi}, \varepsilon) = \lambda\bar{\nu}(\bar{\xi}, \varepsilon/\lambda)$ so (4.7) is equivalent to requiring that

$$\nu_*^\theta(\bar{\xi}) := \limsup_{\varepsilon \to 0} \inf_{w \in \mathcal{W}}\{\varepsilon^{-(1-\theta)} |w|_\mathcal{W} : |\bar{\xi} - \mathbf{T}_o w|_X \leq \varepsilon\}$$

should be bounded for $\bar{\xi} \in \{\mathbf{T}g : g \in \mathcal{G}_*, |g|_\mathcal{V} \text{ bounded}\}$. It is possible to show that ν_*^θ is actually a norm intermediate between the X-norm on $\bar{\mathcal{K}}_o(\theta = 0)$ and the obvious

induced norm: $|\xi|_1 := inf\{|w| : \mathbf{T}w = \xi\}$ on $\mathcal{K}_o(\theta = 1)$. Thus, the condition that $\nu_*^\theta(\mathbf{T}g) < \infty$ is stronger than just requiring $\mathbf{T}g \in \bar{\mathcal{K}}_o$ but is weaker than the exact reachability condition $\mathbf{T}g \in \mathcal{K}_o$ of Theorem 2. We will not analyze ν_*^θ directly but, instead, will use the established theory of interpolation between Banach spaces (cf., e.g., [3]).

THEOREM 5: Assume (H_1) and suppose $\mathcal{K}_\mathbf{F}(\mathcal{W}) \subset \bar{\mathcal{K}}_\mathbf{F} \subset \bar{\mathcal{K}}_o$. Assume (2.12) and suppose that, for some $\theta > \bar{r}$, one has

$$(4.8) \qquad\qquad \mathbf{T}_\mathcal{V}g \in X_\theta \qquad \text{for each } g \in \mathcal{V}$$

where X_θ is an interpolation space $[X_o, X_1]_\theta$ with $X_1 := \mathcal{K}_o$ (with the norm: $|\xi|_1 := inf\{|w|_\mathcal{W} : \mathbf{T}_o w = \xi$ for $\xi \in X_1 = \mathcal{K}_o)$ and $X_o := \bar{\mathcal{K}}_o$ with the X-norm. Then one has the invariance result: $\bar{\mathcal{K}}_o = \bar{\mathcal{K}}_\mathbf{F}$.

REMARK: The hypothesis (4.8) with $\theta > \bar{r}$ is somewhere between taking $\theta = 0$, which just reduces to the (inadequate) hypothesis (H_2), and taking $\theta = 1$ which is equivalent to the exact reachability hypothesis $[R(\mathbf{T}_\mathcal{V}) \subset R(\mathbf{T}_o) = \mathcal{K}_o]$ of Theorem 2. Note that it is easiest to obtain (4.8) if one takes $\mathcal{V}$ as small as possible consistent with (H_1).

PROOF: While there are various possible interpolation functors, the extremal property of the K-functor (see, e.g., Theorem 3.9.1 of [3]) gives a uniform estimate:

$$(4.9) \qquad\qquad s^{-\theta}K(s;\xi) \le C|\xi|_\theta \qquad (s > 0, \xi \in X_\theta)$$

(C depending on the choice of $|\cdot|_\theta$) where

$$\begin{aligned}(4.10) \qquad K(s;\xi) &:= inf\{|\xi_o|_X + s|\xi_1|_1 : \xi_o + \xi_1 = \xi, \xi_1 \in X_1\} \\ &= inf\{s|w|_\mathcal{W} + |\xi - \mathbf{T}_o w| : w \in \mathcal{W}\}.\end{aligned}$$

Fixing $\varepsilon > 0$, define

$$\omega(\nu) = \omega(\nu;\varepsilon) := [C \, \varepsilon^{-(1-\theta)}]^{1/\theta} \, \nu^{1/\theta}.$$

For any $\xi \in X_\theta$ set $\nu := |\xi|_\theta$ and consider $s = \varepsilon/\omega$ in (4.9), (4.10) with $\omega > \omega(\nu)$. From (4.9) this gives $K(s;\xi) < \varepsilon$ so, from (4.10), there exists $w \in \mathcal{W}$ such that

$$|\xi - \mathbf{T}_o w|_X < \varepsilon, \qquad\qquad |w|_\mathcal{W} < \omega.$$

Since we may take ω arbitrarily close to $\omega(\nu)$, this shows:

$$(4.11) \qquad inf\{|w|_\mathcal{W} : |\xi - \mathbf{T}_o w| \le \varepsilon\} \le [C\varepsilon^{-(1-\theta)}]^{1/\theta} \, |\xi|_\theta^{1/\theta}$$

for $\xi \in X_\theta$.

Note that (4.8) implies, by the Closed Graph Theorem, continuity of $\mathbf{T}$ as a linear operator from $\mathcal{V}$ to X_θ, i.e., existence of a constant $\bar{C}$ such that $|\mathbf{T}g|_\theta \le \bar{C}|g|_\mathcal{V}$. Now

fix $\xi_o = \mathbf{T}_o w_o \in \mathcal{K}_o = X_1$ and, letting $\xi = \mathbf{T}g$ in (4.11), note that $|\xi - \mathbf{T}_o w| \le \varepsilon$ if and only if $|\xi_o - [\mathbf{T}g + \mathbf{T}_o(w_o - w)]|_X \le \varepsilon$ so $w' := w_o - w$ is in $\mathcal{C}(g; \xi_o, \varepsilon)$. ¿From (4.11), $\omega > \omega(|\mathbf{T}g|_\theta)$ can be used to estimate w' so

$$
\begin{aligned}
\nu(g; \xi_o, \varepsilon) &= inf\{|w'|_{\mathcal{W}} : w' = w_o - w \in \mathcal{C}(g; \xi_o, \varepsilon)\} \\
&\le |w_o|_{\mathcal{W}} + inf\{|w|_{\mathcal{W}} : w_o - w \in \mathcal{C}(g; \xi_o, \varepsilon)\} \\
&\le |w_o|_{\mathcal{W}} + [C\varepsilon^{-(1-\theta)}]^{1/\theta}[\bar{C}|g|_{\mathcal{V}}]^{1/\theta}
\end{aligned}
$$

(4.12)

We recognize this as (4.7) with $\tilde{r} = 1/\theta$; the assumption $\theta > \bar{r}$ gives $\tilde{r} < 1/\bar{r}$. Thus the Corollary to Theorem 4 applies to show $\mathcal{K}_o \subset \bar{\mathcal{K}}_{\mathbf{F}}$ and one has the desired invariance: $\bar{\mathcal{K}}_{\mathbf{F}} = \bar{\mathcal{K}}_o$. □

COROLLARY: Suppose $\mathcal{V}_o$ is any space for which $\mathbf{T} : \mathcal{V}_o \to X$ is continuous and $\mathcal{V}_1$ is any space for which the exact reachability condition: $\{\mathbf{T}g : g \in \mathcal{V}_1\} \subset \mathcal{K}_o$ holds. Assume (H_1) with $\mathcal{V}$ taken as $\mathcal{V}_\theta := [\mathcal{V}_o, \mathcal{V}_1]_\theta$ for some $\theta > \bar{r}$; assume (3.5) and (2.12). Then one has $\bar{\mathcal{K}}_{\mathbf{F}} = \bar{\mathcal{K}}_o$.

PROOF: Let $\mathbf{T}^o$ be $\mathbf{T} : \mathcal{V}_o \to X_o := \bar{\mathcal{K}}_o$ and let $\mathbf{T}^1$ be $\mathbf{T} : \mathcal{V}_1 \to X_1 := \mathcal{K}_o$; the latter is bounded by the Closed Graph Theorem since $\mathbf{T}\mathcal{V}_1 \subset X_1$. Then interpolation theory [3] gives boundedness of $\mathbf{T}^\theta : \mathcal{V}_\theta \to X_\theta := [X_o, X_1]_\theta$ and Theorem 5 applies. □

A similar but somewhat modified fixpoint argument provides our final result.

THEOREM 6: Assume $(H_1 - i, ii)$ and the continuity (but not necessarily the compactness) of $\mathbf{G}$; assume (3.5). For some $\mathcal{W}'$ as in (4.1) set $\mathcal{G}' := \mathbf{G}\mathcal{W}' := \{\mathbf{G}w : w \in \mathcal{W}'\}$ and assume

(4.13) $\mathcal{K}_o \subset \bar{\mathcal{K}}_g(\mathcal{W}')$ for each $g \in \mathcal{G}'$,

(4.14) $\mathbf{T}\mathcal{G}' := \{\mathbf{T}g : g \in \mathcal{G}'\} = \{\mathbf{T}\mathbf{G}w : w \in \mathcal{W}'\}$ is precompact in X.

Then, one has $cal{K}_{\mathbf{F}} = cal{K}_o$.

PROOF: By Lemma 1 we have $\bar{\mathcal{K}}_{\mathbf{F}} \subset \bar{\mathcal{K}}_o$ and, as above, need only show $\xi \in \bar{\mathcal{K}}_{\mathbf{F}}$ for each $\xi \in \mathcal{K}_o$. Fix $\xi_o \in \mathcal{K}_o$ and note that (4.13) gives $\xi_o - \mathbf{T}\mathcal{G}' := \{\xi_o - \mathbf{T}g : g \in \mathcal{G}'\} \subset \bar{\mathcal{K}}_o(\mathcal{W}')$. Let $X_* := \overline{co}(\xi_o - \mathbf{T}\mathcal{G}') = \xi_o - \overline{co}(\mathbf{T}\mathcal{G}')$ and note that X_* is compact by (4.14) and is contained in $\bar{\mathcal{K}}_o(\mathcal{W}')$ since $\mathcal{W}'$ convex as in (4.1) gives $\bar{\mathcal{K}}_o(\mathcal{W}')$ convex and, of course, closed.

Given any $\varepsilon > 0$, one can find a covering of X_* by ε-balls centered at $\{\xi_j : j = 1, \ldots, J\}$ with each $\xi_j \in \mathcal{K}_o(\mathcal{W}') \cap X_*$ so there exist $w_j \in \mathcal{W}'$ such that $\mathbf{T}_o w_j = \xi_j$. As in the proof of Lemma 4, we can find a continuous partition of unity subordinate to this covering:

$$
\varphi_j \ge 0, \quad \Sigma \varphi_j \equiv 1 \text{ on } X_*, \qquad \varphi_j(\xi) \ne 0 \implies |\xi - \xi_j| < \varepsilon.
$$

and then define $\mathbf{C} = \mathbf{C}_{\xi, \varepsilon}$ by

$$
\mathbf{C}\xi := \Sigma \varphi_j(\xi) w_j,
$$

noting that $\mathbf{C}\xi \in \mathcal{W}'$ for $\xi \in X_*$ by the assumed convexity of $\mathcal{W}'$. Clearly $\mathbf{C} : X_* \to \mathcal{W}'$ is continuous and, as earlier, a simple computation shows that

$$(4.15) \qquad |\xi - \mathbf{T}\mathbf{C}\xi| \leq \varepsilon \quad \text{for} \quad \xi \in X_*.$$

For any $w \in \mathcal{W}'$ we have $[\xi_o - \mathbf{T}\mathbf{G}w] \in X_*]$ so the map:

$$(4.16) \qquad \xi \longmapsto w := \mathbf{C}\xi \longmapsto [\xi_o - \mathbf{T}\mathbf{G}w]$$

is a continuous selfmap of the compact, convex set X_*.

By the Schauder Fixpoint Theorem this map has a fixpoint $\bar{\xi}$ so, setting $\bar{w} := \mathbf{C}\bar{\xi} \in \mathcal{W}'$ we have $\bar{\xi} = \xi_o - \mathbf{T}\mathbf{G}\bar{w}$. Using (4.15), we have

$$|\xi_o - \mathbf{T}_{\mathbf{F}}\bar{w}|_X = |\xi_o - [\mathbf{T}\mathbf{G}\bar{w} + \mathbf{T}\bar{w}]|_X = |\bar{\xi} - \mathbf{T}\bar{w}|_X \leq \varepsilon.$$

Since this is possible for each $\varepsilon > 0$ we have $\xi \in \bar{K}_F(\mathcal{W}') \subset \bar{K}_{\mathbf{F}}$. Since that holds for each $\xi \in K_o$ we have $\bar{K}_o \subset \bar{K}_{\mathbf{F}}$. $\square$

References

[1] J. P. Aubin, Un théorème de compacité. CRAS de Paris, 1963, pp. 5042-5043.

[2] J. P. Aubin and I. Ekeland, Applied Nonlinear Analysis. Wiley-Interscience, New York, 1984.

[3] J. Bergh and J. Lofstrom, Interpolation Spaces, an Introduction. Springer-Verlag, Berlin, 1976.

[4] N. Carmichael and M.D. Quinn, Fixed point methods in nonlinear control. In Distributed Parameter Systems (Lect. Notes in Control and Inf. Sci. 75; edited by F. Kappel, K. Kunisch, W. Schappacher), Springer-Verlag, 1985, pp. 24-51.

[5] J. Diestel and J. J. Uhl, Vector Measures. Amer. Math. Soc., Providence, 1977.

[6] D. Henry, Geometric Theory of Semilinear Parabolic Equations, (Lect. Notes in Math. 840) Springer-Verlag, Berlin, 1981.

[7] K. Naito, Controllability of semilinear control systems, SIAM J. Control/Optim., to appear.

[8] K. Naito, Approximation and controllability for solutions of semilinear control systems, Control-Theory and Advanced Technology 1, 1985, pp. 165-173.

[9] T. I. Seidman, Two compactness lemmas. In Nonlinear Semigroups, Partial Differential Equations, and Attractors (Lecture Notes in Math.; edited by T. P. Gill, W. Zachary), Springer-Verlag, New York, to appear.

[10] T. I. Seidman, Invariance of the reachable set under nonlinear perturbations. SIAM J. Control/Optim., to appear.

[11] T. I. Seidman, Invariance under nonlinear perturbations for reachable and almost-reachable sets. In Proc. IFIP WG-7 Workshop on Control of PDE's (Gainesville, 1986; edited by I. Lasiecka), Springer- Verlag, New York, to appear.

[12] T. I. Seidman, A theorem on convexity. To appear.